Revealing Network Connectivity From Dynamics

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Abstract

We present a method to infer network connectivity from collective dynamics in networks of synchronizing phase oscillators. We study the long-term stationary response to temporally constant driving. For a given driving condition, measuring the phase differences and the collective frequency reveals information about how the oscillators are interconnected. Sufficiently many repetitions for different driving conditions yield the entire network connectivity from measuring the dynamics only. For sparsely connected networks we obtain good predictions of the actual connectivity even for formally under-determined problems.

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Synchronization of networks of coupled units is an ubiquitous phenomenon in nature. It occurs on very different temporal and spatial scales in a variety of systems as different as Josephson junction arrays and networks of neurons in the brain [1, 2, 3]. A central issue in current multidisciplinary research is to understand the relations between network structure and network dynamics. Given an idealized model of the dynamics of the individual units and of their interactions, what can we tell about features of the collective dynamics depending on the network connectivity, say a regular lattice, a random network or some more intricately connected network [2, 3, 4]? For many biological systems, such as networks of neurons, interacting proteins or genes, and ecological foodwebs [9, 10, 11, 12, 13, 14], however, important aspects of the network structure are largely unknown such that inverse methods may prove useful. It would thus be desirable to answer the reverse question: Can we infer information about the connectivity of a network from controlled measurements of its dynamics?

Here we follow this novel perspective for synchronizing phase oscillators that interact on networks of general connectivity. When driving one or more oscillators, the measured phase dynamics reveals information about the specific connectivity. We demonstrate that and how, given a network of N oscillators, each experiment (consisting of driving and measuring) provides N restrictions onto the network connectivity that is defined by N^2 coupling strengths. Exploiting this, we reveal the entire network *connectivity* by repeatedly performing measurements of the *dynamics* only, under N independent driving conditions. Furthermore, assuming that real networks are substantially more sparsely connected than all-to-all, we extend the method to reliably predict the entire connectivity of the network even by a number of experiments that is much smaller than the number of units in the network.

We consider a system of N Kuramoto oscillators, a paradigmatic model that has been successfully used to understand collective dynamical phenomena in engineering, physics, chemistry, biology and medicine [15, 16, 17, 18, 19, 20]. The oscillators are coupled on a directed network of unknown connectivity with their dynamics satisfying

$$\dot{\phi}_i = \omega_{i,0} + \sum_{j=1}^{N} J_{ij} \sin(\phi_j - \phi_i) + I_{i,m} \tag{1}$$

where $\phi_i(t)$ is the phase of oscillator i at time t, $\omega_{i,0}$ its natural frequency and J_{ij} the coupling strength from oscillator j to i ($J_{ij} = 0$ if this connection is absent). The quantity $I_{i,m}$ defines the strength of an external signal to oscillator i for driving condition m; it is identically zero, $I_{i,0} \equiv 0$, if the network is not driven. We define the in-degree $k_i := |\{J_{ij} \neq 0 \,|\, j \in \{1, \dots, N\}\}|$ as the number of incoming links to oscillator i.

Consider the stationary dynamics on a phase locked attractor that is close to in-phase synchrony and thus satisfies $\phi_i(t) - \phi_j(t) = d_{ij}$ where the d_{ij} , $|d_{ij}| \ll 1$, are constant in time. Networks satisfying $J_{ij} \geq 0$ and $|\omega_{i,0} - \omega_{j,0}|$ sufficiently small for all i, j exhibit such a stable phase-locked state close to synchrony. The phase-locked condition for the free (undriven) dynamics reads

$$\Omega_0 = \omega_{i,0} + \sum_{j=1}^{N} J_{ij} \sin(\phi_{j,0} - \phi_{i,0}) \tag{2}$$

where Ω_0 is the collective frequency.

For synchronizing systems, commonly only one or a few scalar quantities (such as one complex order parameter) are computed from measured dynamical data (such as the oscillators' phases), often resulting in a statistically accurate description of the overall network dynamics. Here we take a complementary approach and seek a more detailed description of the network dynamics in order to exploit this information for revealing network connectivity.

We drive one or more oscillators in the network by temporally constant input signals $I_{i,m}$, $i \in \{1, \ldots, N\}$ that can be positive, negative or zero (meaning that oscillator i is not driven). Such inputs effectively change their natural frequencies. Keeping the signal strengths sufficiently small, we structurally perturb the phase-locked state such that it stays phase-locked and close to the original (cf. Fig. 1). Such a driving signal results in a phase pattern of the entire network that depends on the details of the connectivity of that network as well as on the driving signal itself [5, 6, 7, 8, 21, 22]. The perturbed phase-locked state satisfies

$$\Omega_m = \omega_{i,0} + \sum_{j=1}^{N} J_{ij} \sin(\phi_{j,m} - \phi_{i,m}) + I_{i,m} \tag{3}$$

for all $i \in \{1, \ldots, N\}$ where now Ω_m is the new collective frequency that has changed due to the driving and $\phi_{i,m}$ are the stationary phases in response to the driving.

Now take the differences between the phase-locked conditions for the driven and the undriven system,

$$D_{i,m} = \sum_{j=1}^{N} J_{ij} \left[\sin(\phi_{j,m} - \phi_{i,m}) - \sin(\phi_{j,0} - \phi_{i,0})\right] \tag{4}$$

where $D_{i,m} := \Omega_m - \Omega_0 - I_{i,m}$. For sufficiently small structural perturbations we approximate $\sin(x) = x + \mathcal{O}(x^2)$ and abbreviate the phase shifts $\theta_{j,m} := \phi_{j,m} - \phi_{j,0}$ yielding

$$D_{i,m} = \sum_{j=1}^{N} \hat{J}_{ij} \theta_{j,m} \tag{5}$$

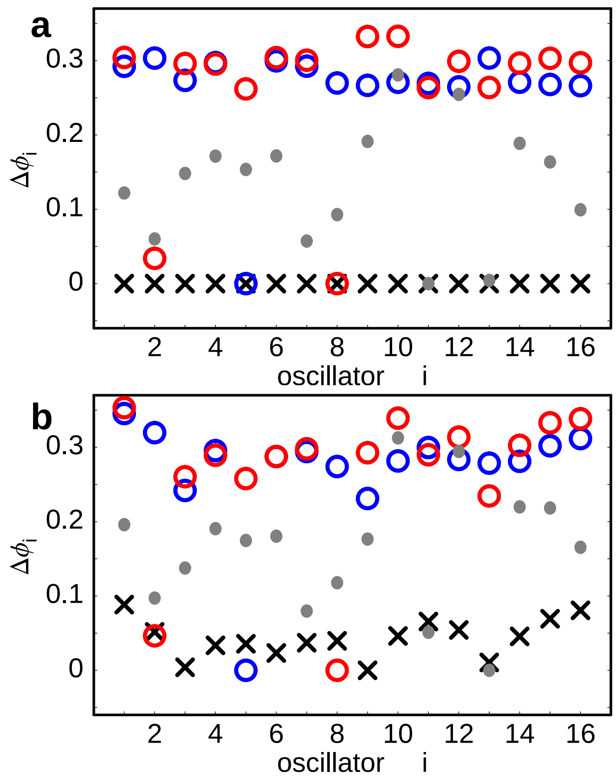

Figure 1: (color) Driving induces phase patterns, implicitly defined by (3). The network has $N = 16$ oscillators, each connected with a constant coupling strength $J_{ij} = 1/k_i$ to $k_i \equiv 8$ randomly selected others ($J_{ij} = 0$ otherwise). (a) Homogeneous frequencies, $\omega_i \equiv 1$; (b) inhomogeneous random frequencies $\omega_i \in [1, 1 + \Delta\omega]$, $\Delta\omega = 0.1$. Both panels display the phase differences $\Delta\phi_i := \max_j\{\phi_j\} - \phi_i$ in the phase-locked states versus i. The responses to three different driving conditions, (blue ◯) one oscillator $i = 5$ driven, $I_{5,1} = 0.3$; (red ◯) two oscillators $i \in \{2, 8\}$ driven, $I_{2,2} = I_{8,2} = 0.3$; (grey •) all oscillators driven by a signal of random strength $I_{i,3} \in [0, 0.3]$ are shown along with the undriven dynamics ($\times$).

where $\hat{J}$ is the $N \times N$ Laplacian matrix is given by

$$\hat{J}_{ij} = \begin{cases} J_{ij} & \text{for } i \neq j \\ -\sum_{k,k\neq j} J_{ik} & \text{for } i = j \end{cases} . \tag{6}$$

Given one driving condition m, we measure $N - 1$ independent phase shifts $\theta_{i,m}$ and one collective frequency Ω_m to obtain N linearized equations (5) that restrict the N^2 dimensional space of all possible network connectivities $(J_{ij})_{i,j\in\{1,\ldots,N\}}$. This is the maximum number of restrictions one can deduce from one experiment. As a consequence, from measurements under linearly inde-

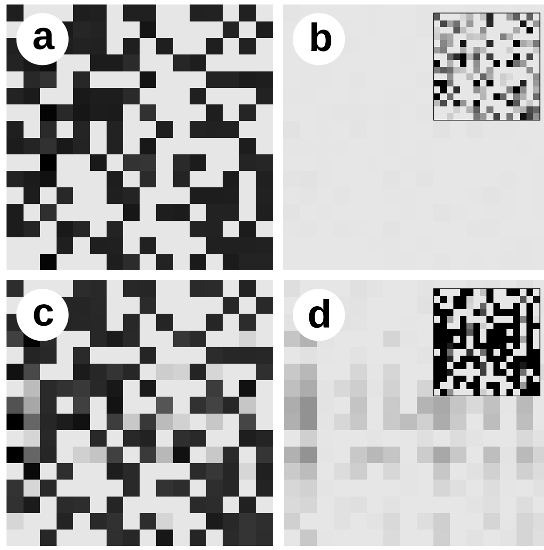

Figure 2: Inferring connectivity from measuring response dynamics. $M = N = 16$ experiments [24]. (a) Connectivity of the network of Fig 1a as obtained using Eqs. (5–7). The matrix of connection strengths J_{ij} is gray-coded from light gray ($J_{ij} = 0$) to black ($J_{ij} = \max_{i',j'}\{J_{i'j'}\}$). (b) Element-wise absolute difference $|J_{ij}^{\text{original}} - J_{ij}^{\text{derived}}|$, plotted on the same scale as (a). Inset shows magnified difference $100 \times |J_{ij}^{\text{original}} - J_{ij}^{\text{derived}}|$ with a cutoff at unity (black). Panels (c) and (d) are analogous to (a) and (b) for the network with inhomogeneous frequencies of Fig. 1b.

pendent driving conditions, we obtain more and more information about the connectivity: After performing M experiments [26] the space of networks is restricted by MN equations

$$D = \hat{J}\theta \tag{7}$$

where $\theta = (\theta_{i,m})_{i\in\{1,\dots,N\},m\in\{1,\dots,M\}}$ is the $N \times M$ matrix of column vectors of phase differences for each experiment m and, analogously, $D = (D_{i,m})_{i,m}$ is the $N \times M$ matrix of the effective frequency offsets. Thus we are left with an $(N - M)N$-dimensional family of possible networks that are consistent with the M measured data sets. In particular, this implies that after $M = N$ experiments the network connectivity is specified completely as given by $\hat{J} = D\theta^{-1}$. We thus find the network *connectivity* from measuring the response *dynamics* only, see, e.g. Fig. 2.

This direct method in principle works for networks with homogeneous as well as with inhomogeneous frequencies [25]. The method is capable of revealing not only which links are present and which are absent but also gives a good quantitative estimate of the actual link strengths J_{ij}. It has, however, also some drawbacks. The problem of solving (7) can be ill-conditioned in the sense that the ratio of the largest and smallest singular value of θ^{T} is large, leading to low-quality reconstruction. Moreover, the direct method might become impractical when studying real-world networks which often consist of a large number N of units and thus would require a large number $M = N$ of (possibly costly) experiments.

Can we obtain the connectivity more efficiently, even with $M < N$ experiments? In many networks, such as networks of neurons in the brain, a substantial number of potential links are *not* present: each node i typically has a number $k_i \ll N$ of links. Here we exploit this fact and look for that connectivity matrix J that has the least number of links (maximum number of $J_{ij} = 0$) but is still consistent with all M measured data sets.

To achieve this goal we use the constraints (7) to parameterize the family of admissible matrices by $(N - M)N$ real parameters P_{ij}, $i \in \{1, \dots, N\}$, $j \in \{M + 1, \dots, N\}$ in a standard way using a singular value decomposition of $\theta^{\mathrm{T}} = USV^{\mathrm{T}}$ where the $M \times N$ matrix S contains the singular values on the diagonal, $S_{ij} = \delta_{ij}\sigma_i \geq 0$. We rewrite the set of all coupling matrices $\hat{J} = DU\tilde{S}V^{\mathrm{T}} + PV$, setting $P_{ij} = 0$ for all $j \leq M$ and $\tilde{S}_{ij} = \delta_{ij}/\sigma_i$ if $\sigma_i > 10^{-4}$ and $\tilde{S}_{ii} = 0$ if $\sigma_i \leq 10^{-4}$. Finally, we minimize the 1-norms of the row vectors of $\hat{J}$ (input coupling strengths)

$$\left\| \hat{J}_i \right\|_1 := \sum_{j=1; j \neq i}^{N} |J_{ij}| \tag{8}$$

with respect to the parameters P, separately for all oscillators i. By this method we find the network J that is closest to the origin $J = 0$, which in 1-norm is one with a minimal number of incoming links (maximal number of zero entries) [27]; thus we find the sparsest of all networks satisfying the measurement data. Reasonably good reconstructions can already be obtained with the number of experiments M being substantially smaller than N, as illustrated in Fig. 3.

How reliable is such a reconstruction? This depends on the details of the network connectivity and the realization of driving. We did a case study for random networks of different sizes N, where each oscillator receives input connections from $k_i \equiv k < N$ randomly chosen others. Using $J_{\text{max}} = \max_{i',j'} \left\{ \left| J_{i'j'}^{\text{derived}} \right|, \left| J_{i'j'}^{\text{original}} \right| \right\}$, define the element-wise relative difference as

$$\Delta J_{ij} := J_{\text{max}}^{-1} \left| J_{ij}^{\text{derived}} - J_{ij}^{\text{original}} \right| \Big/ 2 \tag{9}$$

such that $\Delta J_{ij} \in [0, 1]$. After M experiments, the quality of reconstruction is defined as the fraction

$$Q_\alpha(M) := \frac{1}{N^2} \sum_{i,j} H\left((1 - \alpha) - \Delta J_{ij} \right) \in [0, 1] \tag{10}$$

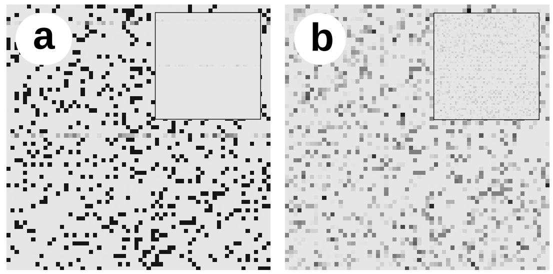

Figure 3: Revealing connectivity with $M < N$ measurements. Network ($N = 64$, $k = 10$, $\Delta\omega = 0$) reconstructed by minimizing the 1-norm, (a) $M = 38$, (b) $M = 24$. The insets show the element-wise absolute differences to the original network.

of coupling strengths which are considered correct. Here $\alpha \leq 1$ is the required accuracy of the coupling strengths and H the Heaviside step function, $H(x) = 1$ for $x \geq 0$, $H(x) = 0$ for $x < 0$. Typically, the quality of reconstruction increases with M (but depends also on the realizations of the experiments), becoming close to one already for M substantially smaller than N, see Fig. 4a. We furthermore evaluated the minimum number of experiments

$$M_{q,\alpha} := \min\{M \,|\, Q_\alpha(M) \geq q\} \tag{11}$$

required for accurate reconstruction on quality level q. Figure 4b shows $M_{0.90,0.95}$, the minimum number of experiments required for having at least $q = 90\%$ of the links accurate in strength on a level of at least $\alpha = 95\%$, as a function of N. The numerics suggests that $M_{q,\alpha}$ generally scales sub-linearly (presumably logarithmically) with network size N for reasonable $0 < 1 - \alpha \ll 1$ and $0 < 1 - q \ll 1$. In particular, it implies that the connectivity of a network can be revealed reliably even if M is much smaller than the network size N.

Recently, the response of synchronizing phase oscillators to different kinds of driving signals has been studied for random networks and lattices [5, 6, 7, 8]. In the present study we took advantage of the fact that in response to controlled driving (cf. also [21, 22]) the dynamics induced may critically depend on the network connectivity (cf. also Fig. 1). This is generally the case if the networks are strongly connected [23] but have otherwise arbitrary connectivities (cf. Eq. (3) and [22]). Thus information about the *connectivity* can be revealed from measuring the response *dynamics*. To achieve this, we exploited all available information of the network dynamics (the $N - 1$ independent phase differences and the collective frequency) rather than only statistical information, such as one complex order parameter. Interestingly, in a recent study, Arenas et al.

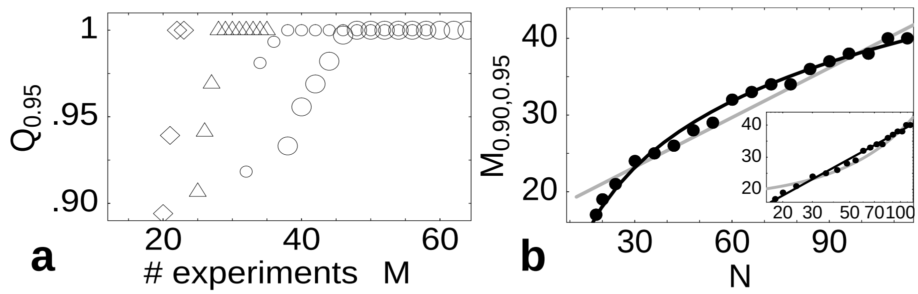

Figure 4: Quality of reconstruction and required number of experiments. (a) Quality of reconstruction ($\alpha = 0.95$) displayed for $k = 10$ and $N = 24$ (◇), $N = 36$ (△), $N = 66$ (○), $N = 96$ (◯) . (b) Minimum number of experiments required ($q = 0.90$, $\alpha = 0.95$) versus network size N with best linear and logarithmic fits (gray and black solid lines). Inset shows same data with N on logarithmic scale.

[20] also used more detailed information of the dynamics and successfully inferred the hierarchical structure of a network.

The method presented here not only identifies where links are present and where they are absent but also gives a good estimate for the strength of each connection. For networks with a substantial number of potential links absent, we furthermore showed how to predict the connectivity in a reliable way even by a number of experiments that is much smaller than the network size. In fact, the numerical evaluation suggests that the number of experiments required for faithful reconstruction only scales sub-linearly with the network size. The relatively simple yet efficient method presented here thus qualifies as potentially practically useful also for real systems of moderate or larger size where the number of measurement might be desired as small as possible. An important question for future research is thus how to extend the method presented here to networks of dynamical elements that are described by more than one variable and that possibly do not synchronize but organize into some other, more complicated, collective state.

The multidisciplinary research community studying networks has recently seen significant progress towards understanding the implications of structural features for network dynamics and function, in particular in biological networks. Interesting examples [9, 10, 11] include (i) network motifs, small subnetworks that occur significantly more often than in randomized networks, have been identified in a variety of complex systems and might be designed for functionality; (ii) a small part of a genetic pathway was successfully identified based on expression profiling; (iii) neural wiring in the brain appears to follow optimization rules. Together with such findings, our results on synchronizing oscillator networks suggest a very promising future direction of research: Methods similar to the one presented here should on the one hand help to better understand structure-dynamics relations from measuring perturbed, possibly complicated, stable dynamics; on the other hand they could also help clarifying structural questions in the first place, e.g., by identifying functionally meaningful parts of a network.

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[24] Here and in the following, the driving signals are independent random $I_{i,m} \in [0, \Delta I]$, where $\Delta I = 0.2$.

[25] The reconstruction quality depends not only on the degree of inhomogeneity, but also, among other parameters, on the realization of it and of the driving.

[26] We discard outliers: they may originate from perturbations that create (i) no structurally stable response or (ii) responses that are close to linearly dependent on others.

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